A Note on the Transmission Line Equation in Terms of Impedance

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INCREASED familiarity derived in handling Maxwell's equations, especially in connection with prohlems arising at very high frequencies, has resulted in a variety of forms for expressing certain laws and behavior. Especially, work hy Schelkunoff in extending the impedance concept[1] shows that impedance can he quite as general and exact a means for expressing electromagnetic relations as are current, voltage, electric and magnetic fields, and vector and scalar potentials.

In reformulating certain prohlems in terms of impedance the content and ultimate solution must of course he equivalent. There may, however, he a considerable change of procedure and sometimes a simplification. For instance, in many cases a single impedance condition can replace the usual two houndary conditions for voltage and current.

One very simple case in which it is perhaps easiest to deal directly with impedance is in the derivation of the transmission line equation on a distrihuted constant basis. In the usual derivation, two linear second order differential equations are ohtained, one for voltage and one for current. The impedance, in terms of which the engineer expresses many of his results, is ohtained as a ratio from solutions for voltage and current. In treating the transmission line from the impedance point of view, without dealing with currents and voltages, a first order non-linear differential equation in terms of impedance and distance is obtained. This impedance equation is a Ricatti equation and could he ohtained from the usual line equations. It is simpler, however, to derive it directly.

As the principal interest of such a treatment lies in the method and in the fact that the line may he tapered, rather than in losses, the derivations will he carried out for lossless lines. Losses can be taken into account hy allowing the inductance per unit length, L, and the capacitance per unit length, C, to become complex quantities.

Consider the section of line dx long, shown in the figure, having an inductance $L\,dx$ and a capacitance $C\,dx$. We can write immediately

$$\begin{aligned} Z_x + dZ &= Z_{x+dx} \\ &= j\omega L\,dx + \frac{1}{j\omega C\,dx + 1/Z_x} \\ &= Z_x + j\omega[L - CZ_x^2]\,dx. \end{aligned} \tag{1}$$

[1] "The Impedance Concept and Its Application to Problems of Reflection, Refraction, Shielding, and Power Absorption," *B.S.T.J.* Vol. 17, pp. 17–48, January, 1938.

Dropping the subscript x, the differential equation for the line in terms of the impedance Z may be written[2]

$$R\frac{dZ}{dx} = j\frac{\omega}{v}(R^2 - Z^2) \tag{2}$$

$$R = (L/C)^{1/2} \tag{3}$$

$$v = (LC)^{-1/2} \tag{4}$$

R is the nominal characteristic impedance, and v is the nominal phase velocity, which is constant for many tapered lines with the same dielectric material separating the conductors throughout their length. In such lines, if the dielectric is air or vacuum, v is c, the velocity of light.

It should not be surprising that (2) is of the first order. Although there are two boundary conditions, the impedances terminating the right and left ends of the line, there are two impedances, that looking toward the right and that looking toward the left. The impedance looking toward the right

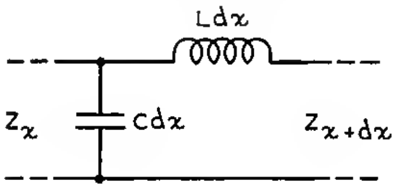

Fig. 1

is unaffected by the left end termination, and that looking toward the left is unaffected by the right end termination.

As R is real, it may be seen from (2) that the only case in which the impedance can equal the nominal characteristic impedance R at all points is for R constant. This tells us that the characteristic impedance of any lossless tapered line is complex. For very gradually tapering lines the characteristic impedance differs from the nominal characteristic impedance principally by a small imaginary component.

The simplest solution of (2) is of course that for a uniform line, with R a constant which will be called R_o. In this case (2) can be integrated directly, giving the familiar result

$$\frac{Z}{R_0} = \tanh(j\omega x/v + K) \tag{5}$$

[2] It is interesting to note that the equation for admittance Y can be obtained by replacing Z by Y and R by $(1/R) = G$ in (2).

Dr. L. A. MacColl has pointed out to the writer that (2) is the same as the electrostatic electron optical equation for paraxial rays. To reduce (2) to the standard form:

$$-\frac{j\omega\, dx}{Rv} = dz \tag{6}$$

$$-R^2 = H(z) \tag{7}$$

$$\frac{dZ}{dz} = H(z) + Z^2 \tag{8}$$

The electron optical equation for paraxial rays is

$$\frac{d\Gamma}{dz} = \frac{3}{16}\left[\frac{V'(z)}{V(z)}\right]^2 + \Gamma^2 \tag{9}$$

$$\Gamma = C - \frac{V'(z)}{4V(z)} \tag{10}$$

Here z is distance along the axis, $V(z)$ is potential on the axis, and C is convergence, or the inverse of the focal distance.

It would seem, then, that from each solution of an electron optical problem, a solution of a tapered line problem could be found, and vice versa.

While it cannot be claimed that anything new has entered the transmission line equation in expressing it in terms of impedance, it does seem that the approach may be stimulating in uncovering hitherto neglected material and analogies.